

MOTION OF CHARGED PARTICLES IN HOMOGENEOUS FIBRATIONS

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ABSTRACT. Let $(M = G/H, g)$ be a reductive homogeneous Riemannian manifold, where g is a G -invariant metric and let T_oM be the tangent space of M at $o = eH$. We study the differential equation $\nabla_{\dot{x}}\dot{x} = kI(\dot{x})$ ($k \in \mathbb{R}$ and I an endomorphism of T_oM), whose solution $x(t)$ represents the motion of a charged particle in M under the electromagnetic field kI . If $k = 0$ then $x(t)$ is a geodesic. We solve such an equation in a Riemannian fibration $K/H \rightarrow G/H \rightarrow G/K$, where G is a Lie group with a bi-invariant Riemannian metric and $H \subset K \subset G$ are closed connected subgroups of G . To this end, we prove a more general result which has its own interest. Namely, we study the motion of a charged particle in a homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ ($\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H respectively), under the following conditions: (i) the tangent space $T_oM \cong \mathfrak{m}$ admits a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ into $\text{Ad}(H)$ -modules, orthogonal with respect to an Ad -invariant inner product of \mathfrak{g} and the G -invariant metric g is diagonal on \mathfrak{m} , (ii) there exist subspaces $\mathfrak{m}_a, \mathfrak{m}_b \subset \mathfrak{m}$ ($a, b = 1, \dots, s$) such that $[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_a$ and (iii) the restriction of the endomorphism I into $\mathfrak{m}_a \oplus \mathfrak{m}_b$ is determined by an element in the center of \mathfrak{h} .

Mathematics Subject Classification. Primary 53C25; Secondary 53C30.

Keywords: Charged particle; electromagnetic field; geodesic; homogeneous space; fibration; generalized flag manifold; Hopf bundle

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $(M = G/H, g)$ be a homogeneous Riemannian manifold, where g is a G -invariant metric with corresponding Levi-Civita connection ∇ . The aim of the present paper is the study of the “charged particle” differential equation

$$\nabla_{\dot{x}}\dot{x} = kI(\dot{x}), \quad (1)$$

where k is some real constant and I an endomorphism of T_oM , the tangent space of M at $o = eH$. Equation (1) appears in a more general context in general relativity as follows ([9]).

Let (M, g) be a Riemannian manifold, F a closed 2-form, and X a vector field on M . We denote by $\iota_X : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M)$ the interior product operator induced by X , and by $\mathcal{L} : TM \rightarrow T^*M$ the Legendre transformation defined by $u \mapsto \mathcal{L}(u)$, $\mathcal{L}(u)(v) = g(u, v)$ ($v \in TM$). A curve $x(t)$ in M is called a *motion of a charged particle under electromagnetic field F* if it satisfies the differential equation

$$\nabla_{\dot{x}}\dot{x} = -\mathcal{L}^{-1}(\iota_{\dot{x}}F),$$

where ∇ is the Levi-Civita connection of M . When $F = 0$ then $x(t)$ is a geodesic in M . In particular, if M is a Kähler manifold with complex structure J there is a natural choice of an electromagnetic field F , namely a scalar multiple of the Kähler form ω , defined by $\omega(X, Y) = g(X, JY)$. Since $-\mathcal{L}^{-1}(\iota_X\omega) = JX$, a curve $x(t)$ is a motion of a charged particle

under electromagnetic field $\kappa\omega$ if and only if $\nabla_{\dot{x}}\dot{x} = kJ(\dot{x})$. We also refer to [7] and [8] for other relevant applications in physics.

Differential equation (1) has been studied by O. Ikawa for various homogeneous spaces (cf. [3], [4], [5], [6]). A class of homogeneous spaces considered in [5] were generalized flag manifolds (or Kähler C-spaces) with two isotropy summands. These spaces were classified by the first author and I. Chrysikos in [1], hence obtaining a concrete class of homogeneous spaces where equation (1) can be solved.

In the present article we solve differential equation (1) for a large class of homogeneous spaces, which are described as follows.

Let $M = G/H$ be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with respect to an Ad-invariant inner product B of \mathfrak{g} and assume that the Lie algebra \mathfrak{h} has non trivial center $\mathfrak{z}(\mathfrak{h})$. The tangent space T_oM at $o = eH$ can be identified to \mathfrak{m} . Let $\pi : G \rightarrow G/H$ be the projection and for $p \in G$, let $\tau_p : G/H \rightarrow G/H$ be the left translation in G/H by p . We assume that the following conditions are satisfied:

(i) The tangent space \mathfrak{m} admits an $\text{Ad}(H)$ -invariant and B -orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s. \quad (2)$$

We endow G/H with the G -invariant Riemannian metric g corresponding to the positive definite inner product

$$\langle \cdot, \cdot \rangle = \lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_s B|_{\mathfrak{m}_s}, \quad \lambda_i > 0 \quad (i = 1, \dots, s), \quad (3)$$

and let ∇ be the corresponding Riemannian connection of g .

(ii) There exist subspaces $\mathfrak{m}_a, \mathfrak{m}_b$, ($a, b = 1, \dots, s$) of \mathfrak{m} such that

$$[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_a. \quad (4)$$

(iii) For $W \in \mathfrak{z}(\mathfrak{h})$ we define the endomorphism $I_o : \mathfrak{m} \rightarrow \mathfrak{m}$ such that

$$I_o|_{\mathfrak{m}_a \oplus \mathfrak{m}_b} = \text{ad}(W)|_{\mathfrak{m}_a} + \frac{1}{\lambda} \text{ad}(W)|_{\mathfrak{m}_b}, \quad \lambda = \frac{\lambda_b}{\lambda_a}. \quad (5)$$

Since $\text{Ad}(h)I_o = I_o \text{Ad}(h)$ for all $h \in H$, we can extend I_o to a G -invariant (1,1)-tensor I on G/H by defining

$$I_{\pi(p)}V_{\pi(p)} := ((\tau_p)_* \circ I_o \circ (\tau_{p^{-1}})_*)V_{\pi(p)}, \quad V_{\pi(p)} \in T_{\pi(p)}(G/H). \quad (6)$$

We denote a homogeneous space satisfying conditions (i) – (iii) by $(G/H, g, I, \lambda)$.

Definition 1.1. For k a real constant a curve $x(t)$ is called a motion of a charged particle under the electromagnetic field kI if it is a solution of the differential equation

$$\nabla_{\dot{x}}\dot{x} = kI(\dot{x}). \quad (7)$$

Note that if $k = 0$ then $x(t)$ is a geodesic.

We prove the following:

Theorem 1.2. *Let $(G/H, g, I, \lambda)$ be a Riemannian homogeneous space satisfying conditions (i), (ii) and (iii). Let $x(t)$ be the motion of a charged particle given by (7) with initial conditions*

$$x(0) = o \quad \text{and} \quad \dot{x}(0) = X_a + X_b, \quad (8)$$

where $X_a \in \mathfrak{m}_a, X_b \in \mathfrak{m}_b$. Then the curve $x(t)$ is given by

$$x(t) = \exp t(X_a + \lambda X_b + kW) \exp t(1 - \lambda)(X_b + \frac{k}{\lambda}W) \cdot o. \quad (9)$$

The above Theorem generalizes Ikawa's result.

As a consequence, we obtain the following description of corresponding motion in homogeneous fibrations:

Theorem 1.3. *Let G be a Lie group admitting a bi-invariant Riemannian metric and let B be the corresponding Ad-invariant positive definite inner product on \mathfrak{g} . Let K, H be closed and connected subgroups of G , such that $H \subset K \subset G$ and such that the Lie algebra of H has non trivial center. We identify the tangent spaces $T_o(G/H), T_o(G/K)$ and $T_o(K/H)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$ and \mathfrak{m}_2 of \mathfrak{g} , such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We endow G/H with a G -invariant Riemannian metric g_λ corresponding to the $\text{Ad}(H)$ -invariant positive definite inner product*

$$\langle \cdot, \cdot \rangle = B|_{\mathfrak{m}_1} + \lambda B|_{\mathfrak{m}_2}, \quad \lambda > 0,$$

on \mathfrak{m} . Moreover, for $W \in \mathfrak{z}(\mathfrak{h})$, let I^W be the G -invariant $(1, 1)$ -tensor on G/H such that $I_o^W = \text{ad}(W)|_{\mathfrak{m}_1} + \frac{1}{\lambda} \text{ad}(W)|_{\mathfrak{m}_2}$. Let $X = X_1 + X_2 \in \mathfrak{m}$ with $X_i \in \mathfrak{m}_i, i = 1, 2$. Then the motion of a charged particle in G/H under electromagnetic field kI with initial conditions $x(0) = o$ and $\dot{x}(0) = X_1 + X_2$ is the curve $x : \mathbb{R} \rightarrow G/H$ given by

$$x(t) = \exp t(X_1 + \lambda X_2 + kW) \exp t(1 - \lambda)(X_2 + \frac{k}{\lambda}W) \cdot o. \quad (10)$$

We will prove Theorems 1.2 and 1.3 in Section 2. In Section 3 we will give examples of homogeneous spaces satisfying the conditions of Theorem 1.3.

Acknowledgements. The work was supported by Grant #E.037 from the Research Committee of the University of Patras (Programme K. Karatheodori).

2. PROOF OF THE MAIN RESULTS

We need to show that

$$g(\nabla_{\dot{x}} \dot{x}, V) = g(kI(\dot{x}), V), \quad (11)$$

for any vector field V in G/H . By using Koszul's formula the left-hand side of (11) is given by

$$g(V, \nabla_{\dot{x}} \dot{x}) = \dot{x}g(V, \dot{x}) + g(\dot{x}, [V, \dot{x}]) - \frac{1}{2}Vg(\dot{x}, \dot{x}). \quad (12)$$

We set $X = X_a + \lambda X_b + kW$, $Y = (1 - \lambda)(X_b + \frac{k}{\lambda}W)$, and $\alpha : \mathbb{R} \rightarrow G$ with $\alpha(t) = \exp tX \exp tY$ so $x = \pi \circ \alpha$. We also consider the one-parameter family of automorphisms

$$T : \mathbb{R} \rightarrow \text{Aut}(G) \quad \text{with} \quad T(t) = \text{Ad}(\exp(-tY)). \quad (13)$$

We will need the following:

Lemma 2.1. *The following relations are satisfied:*

- 1) $T(t)X_a \in \mathfrak{m}_a$, $t \in \mathbb{R}$,
- 2) $\dot{x}(t) = (\tau_{\alpha(t)})_*(T(t)X_a + X_b)$.

Proof. To prove 1) we recall that

$$T(t)X_a = \sum_{n=0}^{\infty} \frac{t^n}{n!} \text{ad}^n(-Y)X_a. \quad (14)$$

Moreover, by taking into account relation (4) and the $\text{ad}(\mathfrak{h})$ -invariance of the subspace \mathfrak{m}_a we deduce that $[Y, \mathfrak{m}_a] \subset \mathfrak{m}_a$. Therefore, for any $N \in \mathbb{N}$, it is $\sum_{n=0}^N \frac{t^n}{n!} \text{ad}^n(-Y)X_a \in \mathfrak{m}_a$, so by taking the limit $N \rightarrow \infty$ in (14) we obtain that $T(t)X_a \in \mathfrak{m}_a$.

We now prove 2). For $p \in G$, let L_p, R_p be the left and right translations respectively in G by p . We compute:

$$\begin{aligned} \dot{x}(t) &= \pi_*(\dot{\alpha}(t)) = \pi_*\left(\frac{d}{ds}\Big|_{s=0} \alpha(t+s)\right) = \pi_*\left(\frac{d}{ds}\Big|_{s=0} \exp(t+s)X \exp(t+s)Y\right) \\ &= \pi_*\left(\frac{d}{ds}\Big|_{s=0} \exp(t+s)X \exp tY + \frac{d}{ds}\Big|_{s=0} \exp tX \exp(t+s)Y\right) \\ &= \pi_*\left(\frac{d}{ds}\Big|_{s=0} \exp sX \exp tX \exp tY + \frac{d}{ds}\Big|_{s=0} \exp tX \exp tY \exp sY\right) \\ &= \pi_*\left(\frac{d}{ds}\Big|_{s=0} \exp sX \alpha(t) + \frac{d}{ds}\Big|_{s=0} \alpha(t) \exp sY\right) = \pi_*((R_{\alpha(t)})_*(X) + (L_{\alpha(t)})_*Y) \\ &= (\pi \circ L_{\alpha(t)} \circ L_{\alpha(t)^{-1}})_*((R_{\alpha(t)})_*(X) + (L_{\alpha(t)})_*Y) = (\pi \circ L_{\alpha(t)})_*(\text{Ad}(\alpha(t)^{-1})X + Y) \\ &= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\alpha(t)^{-1})X + Y) = (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY) \exp(-tX))X + Y) \\ &= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X + Y) \\ &= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + \text{Ad}(\exp(-tY))(\lambda X_b + kW) + Y) \\ &= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + \text{Ad}(\exp(-tY))(\frac{\lambda}{1-\lambda}Y) + Y) \\ &= (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + \frac{\lambda}{1-\lambda}Y + Y) = (\tau_{\alpha(t)} \circ \pi)_*(\text{Ad}(\exp(-tY))X_a + X_b + \frac{k}{\lambda}W) \\ &= (\tau_{\alpha(t)})_*(\text{Ad}(\exp(-tY))X_a + X_b) = (\tau_{\alpha(t)})_*(T(t)X_a + X_b). \end{aligned}$$

□

For any $p \in G$ we observe that the vector field

$$\pi(p) \mapsto (\tau_p \circ \pi)_*(\text{Ad}(p^{-1})X + Y), \quad (15)$$

is a well defined local extension of $\dot{x}(t)$ in G/H , therefore, the vector field $\nabla_{\dot{x}}\dot{x}$ is well defined. Moreover, in view of equation (12), since $T_{\pi(p)}(G/K)$ admits a basis of left invariant vectors, it suffices to take V as

$$V_{\pi(p)} = (\tau_p)_*Z, \quad Z \in \mathfrak{m}. \quad (16)$$

Proof of Theorem 1.2.

We will first simplify each term of the right-hand side of equation (12). By using Lemma 2.1, equations (16) and (3) as well as the G -invariance of the metric g , the first term of the right-hand side of (12) becomes

$$\begin{aligned} \dot{x}(t)g(V, \dot{x}) &= \left. \frac{d}{ds} \right|_{s=0} g((\tau_{\alpha(t+s)})_*Z, (\tau_{\alpha(t+s)})_*(T(t+s)X_a + X_b)) \\ &= \left. \frac{d}{ds} \right|_{s=0} \langle Z, T(t+s)X_a + X_b \rangle = \left. \frac{d}{ds} \right|_{s=0} \langle Z, T(s)T(t)X_a + X_b \rangle \\ &= \langle Z, [T(t)X_a, Y] \rangle = (1 - \lambda) \langle Z, [T(t)X_a, X_b + \frac{k}{\lambda}W] \rangle \\ &= (1 - \lambda) \lambda_a B(Z, [T(t)X_a, X_b + \frac{k}{\lambda}W]) \\ &= (\lambda_a - \lambda_b) B(Z, [T(t)X_a, X_b + \frac{k}{\lambda}W]). \end{aligned} \quad (17)$$

Similarly, and by using the B -orthogonality of \mathfrak{h} and \mathfrak{m} as well as the Ad -invariance of B , the second term of the right-hand side of (12) becomes

$$\begin{aligned} g(\dot{x}(t), [V, \dot{x}]_{x(t)}) &= g((\tau_{\alpha(t)})_*(T(t)X_a + X_b), [(\tau_{\alpha(t)})_*Z, (\tau_{\alpha(t)})_*(T(t)X_a + X_b)]_{\mathfrak{m}}) \\ &= g((\tau_{\alpha(t)})_*(T(t)X_a + X_b), (\tau_{\alpha(t)})_*[Z, T(t)X_a + X_b]_{\mathfrak{m}}) \\ &= \langle T(t)X_a + X_b, [Z, T(t)X_a + X_b]_{\mathfrak{m}} \rangle \\ &= \lambda_a B(T(t)X_a, [Z, T(t)X_a]_{\mathfrak{m}}) + \lambda_a B(T(t)X_a, [Z, X_b]_{\mathfrak{m}}) \\ &\quad + \lambda_b B(X_b, [Z, T(t)X_a]_{\mathfrak{m}}) + \lambda_b B(X_b, [Z, X_b]_{\mathfrak{m}}) \\ &= \lambda_a B(T(t)X_a, [Z, T(t)X_a]) + \lambda_a B(T(t)X_a, [Z, X_b]) \\ &\quad + \lambda_b B(X_b, [Z, T(t)X_a]) + \lambda_b B(X_b, [Z, X_b]) \\ &= (\lambda_b - \lambda_a) B(Z, [T(t)X_a, X_b]), \end{aligned} \quad (18)$$

where for the last equality we used the Ad -invariance of B .

Finally, for the third term of the right hand side of equation (12), we use the local extension (15) of \dot{x} as well as (16). We have that

$$\begin{aligned} V_{x(t)}g(\dot{x}, \dot{x}) &= ((\tau_{\alpha(t)})_*Z)g(\dot{x}, \dot{x}) \\ &= \left. \frac{d}{ds} \right|_{s=0} g(((\tau_{p^{-1}})_* \circ \pi_*)(\text{Ad}(p^{-1})X + Y), ((\tau_{p^{-1}})_* \circ \pi_*)(\text{Ad}(p^{-1})X + Y)) \\ &= \langle \pi_*(\text{Ad}(p^{-1})X + Y), \pi_*(\text{Ad}(p^{-1})X + Y) \rangle, \end{aligned} \quad (19)$$

where $p = \alpha(t) \exp sZ$. Moreover, we have that

$$\begin{aligned}
\left. \frac{d}{ds} \right|_{s=0} \pi_*(\text{Ad}(p^{-1})X) &= \left. \frac{d}{ds} \right|_{s=0} \pi_*(\text{Ad}(\exp(-sZ))T(t)X) \\
&= \pi_* \left(\left. \frac{d}{ds} \right|_{s=0} (\text{Ad}(\exp(-sZ))T(t)X) \right) \\
&= [T(t)X, Z]_{\mathfrak{m}} = [T(t)X_a + \lambda X_b + \frac{k}{\lambda}W, Z]_{\mathfrak{m}}. \tag{20}
\end{aligned}$$

By substituting equation (20) into equation (19) and by using Lemma 2.1 and the Ad-invariance of B we obtain that

$$\begin{aligned}
\frac{1}{2}V_{x(t)}g(\dot{x}, \dot{x}) &= -\langle [T(t)X_a + \lambda X_b + \frac{k}{\lambda}W, Z]_{\mathfrak{m}}, T(t)X_a + X_b \rangle \\
&= -\langle [T(t)X_a, Z]_{\mathfrak{m}}, T(t)X_a \rangle - \langle [T(t)X_a, Z]_{\mathfrak{m}}, X_b \rangle - \lambda \langle [X_b, Z]_{\mathfrak{m}}, T(t)X_a \rangle \\
&\quad - \lambda \langle [X_b, Z]_{\mathfrak{m}}, X_b \rangle - \frac{k}{\lambda} \langle [W, Z]_{\mathfrak{m}}, T(t)X_a \rangle - \frac{k}{\lambda} \langle [W, Z]_{\mathfrak{m}}, X_b \rangle \\
&= -\lambda_a B([T(t)X_a, Z]_{\mathfrak{m}}, T(t)X_a) - \lambda_b B([T(t)X_a, Z]_{\mathfrak{m}}, X_b) - \lambda \lambda_a B([X_b, Z]_{\mathfrak{m}}, T(t)X_a) \\
&\quad - \lambda \lambda_b B([X_b, Z]_{\mathfrak{m}}, X_b) - \frac{k}{\lambda} \lambda_a B([W, Z]_{\mathfrak{m}}, T(t)X_a) - \frac{k}{\lambda} \lambda_b B([W, Z]_{\mathfrak{m}}, X_b) \\
&= -\lambda_a B([T(t)X_a, Z], T(t)X_a) - \lambda_b B([T(t)X_a, Z], X_b) - \lambda \lambda_a B([X_b, Z], T(t)X_a) \\
&\quad - \lambda \lambda_b B([X_b, Z], X_b) - \frac{k}{\lambda} \lambda_a B([W, Z], T(t)X_a) - \frac{k}{\lambda} \lambda_b B([W, Z], X_b) \\
&= -\frac{k}{\lambda} \lambda_a B([W, Z], T(t)X_a) - \frac{k}{\lambda} \lambda_b B([W, Z], X_b). \\
&= -\frac{k}{\lambda} \lambda_a B(Z, [T(t)X_a, W]) - \frac{k}{\lambda} \lambda_b B(Z, [X_b, W]). \tag{21}
\end{aligned}$$

By substituting equations (17), (18) and (21) in equation (12) and by adding these we obtain that

$$g(V, \nabla_{\dot{x}} \dot{x}) = -k \lambda_a B(Z, [TX_a + X_b, W]). \tag{22}$$

Next, we simplify the right-hand side of equation (11). By using relations (3), (5) and (6), Lemma 2.1, the G -invariance of g and the Ad-invariance of B , we obtain that

$$\begin{aligned}
g(kI(\dot{x}), V) &= kg(((\tau_{\alpha(t)})_* \circ I_o)(T(t)X_a + X_b), (\tau_{\alpha(t)})_* Z) \\
&= k \langle I_o(T(t)X_a + X_b), Z \rangle = k \langle [W, T(t)X_a], Z \rangle + \frac{k}{\lambda} \langle [W, X_b], Z \rangle \\
&= k \lambda_a B([W, T(t)X_a], Z) + \frac{k}{\lambda} \lambda_b B([W, X_b], Z) \\
&= -k \lambda_a B(Z, [TX_a + X_b, W]),
\end{aligned}$$

which, by equation (22) is equal to $g(V, \nabla_{\dot{x}} \dot{x})$. Therefore, relation (11) holds and the proof of Theorem 1.2 is completed. \square

Proof of Theorem 1.3

Let $\mathfrak{h}, \mathfrak{k}, \mathfrak{g}$ be the Lie algebras of the groups H, K, G respectively. The subspaces \mathfrak{m}_1 and \mathfrak{m}_2 can be obtained from the B -orthogonal decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_1 \quad \text{and} \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_2, \quad (23)$$

such that

$$\text{Ad}(K)\mathfrak{m}_1 \subset \mathfrak{m}_1, \quad \text{Ad}(H)\mathfrak{m}_2 \subset \mathfrak{m}_2. \quad (24)$$

Therefore, the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is $\text{Ad}(H)$ -invariant and B -orthogonal. Moreover, since $\text{Ad}(K)\mathfrak{m}_1 \subset \mathfrak{m}_1$, we have that $[\mathfrak{m}_1, \mathfrak{k}] \subset \mathfrak{m}_1$, therefore,

$$[\mathfrak{m}_1, \mathfrak{m}_2] \subset [\mathfrak{m}_1, \mathfrak{k}] \subset \mathfrak{m}_1. \quad (25)$$

By taking into account relation (25), it is straightforward to check that the space $(G/H, g_\lambda, I^W, \lambda)$ satisfies the conditions of Theorem 1.2, and this completes the proof.

3. EXAMPLES

3.1. Lie groups. Let G be a Lie group admitting a bi-invariant Riemannian metric g . Let K be a connected subgroup of G and let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K respectively. The bi-invariant metric g corresponds to an Ad -invariant positive definite inner product B on \mathfrak{g} , which induces an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This decomposition is $\text{Ad}(K)$ -invariant. Indeed, for any $X_{\mathfrak{k}}, Y_{\mathfrak{k}} \in \mathfrak{k}$ and $X_{\mathfrak{m}} \in \mathfrak{m}$, we set $Z_{\mathfrak{k}} = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] \in \mathfrak{k}$. Then

$$B([X_{\mathfrak{k}}, X_{\mathfrak{m}}], Y_{\mathfrak{k}}) = -B(X_{\mathfrak{m}}, [X_{\mathfrak{k}}, Y_{\mathfrak{k}}]) = -B(X_{\mathfrak{m}}, Z_{\mathfrak{k}}) = 0.$$

It follows that $[X_{\mathfrak{k}}, X_{\mathfrak{m}}] \in \mathfrak{m}$, and since K is connected then $\text{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$, which in turn gives that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

We view G as the homogeneous space $G/\{e\}$ and we endow G with the left invariant metric g_λ corresponding to the positive definite inner product

$$\langle \cdot, \cdot \rangle = B|_{\mathfrak{m}} + \lambda B|_{\mathfrak{k}}, \quad \lambda > 0. \quad (26)$$

By taking $\mathfrak{m}_1 = \mathfrak{m}$ and $\mathfrak{m}_2 = \mathfrak{k}$ in Theorem 1.3, then we obtain the motion of a charged particle in a Lie group G .

3.2. Ikawa's result. Let $M = G/H$ be a homogeneous space satisfying the conditions of Theorem 1.2 with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then we obtain Ikawa's result [5, Theorem 1.1] under the only condition $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

3.3. Hopf bundles. Let $G = \text{U}(n+1)$, $H = \text{U}(n)$ and $K = \text{U}(n) \times \text{U}(1)$. Then the fibration $K/H \rightarrow G/H \rightarrow G/K$ is the homogeneous Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^{2n+1} \rightarrow \mathbb{C}P^n. \quad (27)$$

Since $\text{U}(n+1)$ is compact, it admits a bi-invariant metric corresponding to an $\text{Ad}(\text{U}(n+1))$ -invariant positive definite inner product B on $\mathfrak{u}(n+1)$. We identify each of the spaces $T_o\mathbb{S}^{2n+1} = T_o(G/H)$, $T_o\mathbb{C}P^n = T_o(G/K)$, and $T_o\mathbb{S}^1 = T_o(K/H)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$, and \mathfrak{m}_2 of $\mathfrak{u}(n+1)$. Consider the one parameter family of metrics g_λ on \mathbb{S}^{2n+1} corresponding to the positive definite inner products

$$\langle \cdot, \cdot \rangle = B|_{\mathfrak{m}_1} + \lambda B|_{\mathfrak{m}_2}, \quad \lambda > 0 \quad (28)$$

on $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Note that for $\lambda = 1$ the inner product (28) gives the standard metric g_1 on \mathbb{S}^{2n+1} . Then the hypotheses of Theorem 1.3 are satisfied, hence the curve (10) describes the motion of a charged particle in $(\mathbb{S}^{2n+1}, g_\lambda)$. However, it is known ([Ya]) that $G/H = \mathrm{U}(n+1)/\mathrm{U}(n)$ is a weakly symmetric space, hence in this case it can be shown that motion (10) reduces to $x(t) = \exp t(a + X_1 + X_2 + kW) \cdot o$, $X_i \in \mathfrak{m}_i$ ($i = 1, 2$) and $a \in \mathfrak{h}$, which depends on X_1, X_2 .

3.4. Twistor fibrations. For G semisimple let $F = G/H$ be a generalised flag manifold, i.e. an adjoint orbit of an element W in \mathfrak{g} . It is known ([2]) that any flag manifold $F = G/H$ can be fibered over a compact inner symmetric space G/K ($H \subset K$) under the twistor fibration $\pi : G/H \rightarrow G/K$. The *normal metric* of G/H is the G -invariant metric induced by the negative of the Killing form \mathfrak{g} , denoted by B . We endow G/H with the “deformation” of the normal metric along the fibers K/H of the twistor fibration, given by

$$\langle \cdot, \cdot \rangle = B|_{T_o(G/K)} + \lambda B|_{T_o(K/H)}, \quad \lambda > 0.$$

Then, by virtue of Theorem 1.3, the equation of motion of a charged particle in $(F, \langle \cdot, \cdot \rangle)$ under electromagnetic field kI , is given by (10).

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